A dynamic correspondence between Bose-Einstein condensates and Friedmann-Lemaître-Robertson-Walker and Bianchi I cosmology with a cosmological constant

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In some interesting work of James Lidsey, the dynamics of Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology with positive curvature and a perfect fluid matter source is shown to be modeled in terms of a time-dependent, harmonically trapped Bose-Einstein condensate. In the present work, we extend this dynamic correspondence to both FLRW and Bianchi I cosmologies in arbitrary dimension, especially when a cosmological constant is present.

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I. INTRODUCTION

The general feature of this paper is a connection between a non-gravitational system and a gravitational system. Such connections of course are of growing interest and importance. More specifically, extending methods initiated in [12], we set up a correspondence between Bose-Einstein condensates governed by a time-dependent, harmonic trapping potential and both Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology and Bianchi I cosmology, in an arbitrary dimension, and with a non-zero cosmological constant in both cases. The correspondence is presented by way of two tables (Table I and Table II) that match cosmological parameters (scale factors, scalar pressure and energy density, Hubble parameter) with wavepacket parameters given in terms of the harmonic trapping frequency $\omega(t)$ and *moments* $I_j(t), j = 2, 3, 4$ (with $I_2(t) > 0$) of the wavefunction of the Gross-Pitaevskii equation - under the assumption that the atomic interaction parameter is a constant in time. Here t is "laboratory time" that one passes to from cosmic time τ in the Einstein field equations. The moments satisfy the conservation law

$$\lambda(t) \stackrel{def}{=} 2I_2(t)I_4(t) - I_3^2(t)/4 = \text{ a constant } \lambda. \tag{1}$$

Moreover, $I_2(t)$ gives rise to a solution $X(t) \stackrel{def.}{=} I_2^{1/2}(t)$ of the classical Ermakov-Milne-Pinney (EMP) equation [13]

$$\frac{d^2X}{dt^2} + \omega^2(t)X = \frac{\lambda}{X^3}. \tag{2}$$

On the other hand, one knows also that the Einstein equations for a FLRW universe, and even for the anisotropic Bianchi I and Bianchi V universes, admit a formulation in terms of a suitable EMP equation, of classical or of generalized type [5–8, 11, 15, 16]. This information coupled with equation (2) allows one to proceed in setting up the desired correspondence - at least in the FLRW and Bianchi I cases.

For certain cosmological models, for example a stiff fluid model, the frequency $\omega(t)$ can be determined, and thus the external potential is explicated. A new issue arises however when a non-vanishing cosmological constant is present. Our analysis shows that in this case one must employ elliptic functions to solve the appropriate moment equations that arise - a matter discussed in section V.

As indicated in [12], through the condensed matter - cosmology correspondence via EMP equations (as considered here) there is the increased possibility for insight into the hidden symmetries of these systems. Further connections between non-gravitational and gravitational systems are explored in [4], for example.

II. AN EMP FORMULATION OF FLRW AND BIANCHI I

We begin by formulating the $d-$dimensional FLRW and Bianchi I equations with a cosmological constant Λ_d (for $d \geq 3$) as a single EMP equation of classical type; we do this (for the record) as it has not been done before in

arbitrary dimensions, even though the extension is straightforward. The classical EMP equation is of the form

$$\ddot{Y} + A(t)Y = \frac{\mu}{Y^3} \tag{3}$$

for $\ddot{Y} = \frac{d^2Y}{dt^2}$, and for some constant μ; compare equation (2).

We consider a scalar field ϕ and a potential V, and let $a(\tau)$ denote the scale factor, in which case the FLRW equations assume the following form, for the Hubble parameter $H \overset{def.}{=} \frac{a'}{a}$:

$$H^2 + \frac{k}{a^2} = \frac{2\Lambda_d}{(d-1)(d-2)} + \frac{2K_d}{(d-1)(d-2)}\left[\frac{(\phi')^2}{2} + V\circ\phi + \frac{D}{a^n}\right], \tag{4}$$

$$\phi'\phi'' + (d-1)H(\phi')^2 + (V'\circ\phi)\phi' = 0, \tag{5}$$

with $k = 0, -1$, or 1 the curvature parameter, $K_d \overset{def.}{=} 8\pi G_d$ for $G_d =$ the gravitational constant and Λ_d the cosmological constant.

Suppose $f(t) > 0$ is a function with inverse function $T(\tau)$ (i.e. $f(T(\tau)) = \tau, T(f(t)) = t$) such that $T'(\tau) = a(\tau)$. Define

$$Y(t) \overset{def.}{=} a(f(t)), \phi_1(t) \overset{def.}{=} \phi(f(t)). \tag{6}$$

The method of [11] shows immediately that equations (4) and (5) lead to the EMP equation

$$\ddot{Y}(t) + \frac{nK_d}{2(d-2)}\dot{\phi}_1(t)^2 Y(t) = \frac{k}{Y(t)^3}. \tag{7}$$

For the anisotropic $d-$dimensional Bianchi I cosmological model with metric $ds^2 = -d\tau^2 + X_1(\tau)^2dx_1^2 + \cdots + X_{d-1}(\tau)^2dx_{d-1}^2$, the field equations take the form

$$\sum_{l<k} H_l H_k = K_d\left[\frac{(\phi')^2}{2} + V\circ\phi\right] + \Lambda_d, \tag{8}$$

$$\sum_{l\neq i}\left(H_l' + H_l^2\right) + \sum_{\substack{l<k\\ l,k\neq i}} H_l H_k = -K_d\left[\frac{(\phi')^2}{2} - V\circ\phi\right] + \Lambda_d \tag{9}$$

where $i, l, k \in \{1, \ldots, d-1\}$ and $H_l \overset{def.}{=} \dot{X}_l/X_l$. By making the substitution $X_l(\tau) = R(\tau)e^{\alpha_l(\tau)}$ for functions $R(\tau) > 0$ and $\alpha_l(\tau)$ satisfying $\alpha_1(\tau) + \cdots + \alpha_{d-1}(\tau) = 0$, one sees that the field equations (8) and (9) can be written as

$$\frac{(d-1)(d-2)}{2}H_R^2 - \frac{DK_d}{R^{2(d-1)}} = K_d\left[\frac{(\phi')^2}{2} + V\circ\phi\right] + \Lambda_d, \tag{10}$$

$$(d-2)H_R' + \frac{(d-1)(d-2)}{2}H_R^2 + \frac{DK_d}{R^{2(d-1)}} = -K_d\left[\frac{(\phi')^2}{2} - V\circ\phi\right] + \Lambda_d \tag{11}$$

since $R = (X_1X_2\cdots X_{d-1})^{1/(d-1)}$, where $H_R(\tau) \overset{def.}{=} R'(\tau)/R(\tau)$, $D \overset{def.}{=} \frac{R^{2(d-1)}}{2(d-1)K_d}\sum_{l<k}(H_l - H_k)^2$. D is a constant quantity by a simple lemma which states that for any differentiable function $g(\tau)$, any positive differentiable function $R(\tau)$, and $M \in \mathbb{R}$, the function $g(\tau)R(\tau)^M$ is constant if and only if $g'(\tau) + Mg(\tau)\frac{R'(\tau)}{R(\tau)} = 0$. By equating the left-hand sides of any two Einstein equations (9), indexed by $i \neq j$, we see that the latter equation holds for $g = H_i - H_j$ and $M = (d-1)$, so that $(H_i - H_j)R^{d-1}$ is constant for all $i, j \in \{1, \ldots, d-1\}$.

Similar to the above argument, by taking $f(t) > 0$ to be the inverse of $T(\tau)$ such that $T'(\tau) = R(\tau)^{(d-1)}$, and by defining $Y(t) \overset{def.}{=} R(f(t))^{(d-1)}, \phi_1(t) \overset{def.}{=} \phi(f(t))$, we can show that the Bianchi I field equations (10), (11) lead to the classical EMP

$$\ddot{Y}(t) + \frac{(d-1)K_d}{(d-2)}\dot{\phi}_1(t)^2 Y(t) = \frac{-2(d-1)K_dD}{(d-2)Y(t)^3}. \tag{12}$$

III. ELLIPTIC FUNCTIONS INTERLUDE

In the case of a non-vanishing cosmological constant, that we shall give attention to, the following differential equation (which is of some independent interest)

$$\frac{\dot{y}^2}{4} = \frac{2A}{y^2} - B + Cy \tag{13}$$

arises, where $A > 0, C \neq 0$; see equation (35). Its solutions involve elliptic functions. Before setting up the dynamic correspondence between cosmological and condensate systems we present a brief interlude regarding equation (13), which in particular will allow, conveniently, for the introduction of some notation needed later.

We shall need the elliptic functions $EF(x,k), E(u,k)$ of the *first* and *second* kind, respectively, with modulus k [2, 3, 9, 10, 14] given by

$$\begin{aligned} EF(x,k) &\stackrel{def.}{=} \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \\ E(u,k) &\stackrel{def.}{=} \int_0^u dn^2(v,k)dv = \int_0^{sn(u,k)} \sqrt{\frac{1-k^2t^2}{1-t^2}}dt. \end{aligned} \tag{14}$$

Sometimes, as is usual, the modulus k is suppressed in the notation and one writes $E(u), sn(u)$, for example, for $E(u,k), sn(u,k)$. Given that $C \neq 0$, we can define constants $m = m(C), n = n(B,C)$ and Weierstrass invariants $g_2 = g_2(B,C), g_3 = g_3(A,B,C), p = p(B,C), q = q(A,B,C)$ by

$$\begin{aligned} m &\stackrel{def.}{=} \frac{C}{|C|}e^{\frac{1}{3}\log\frac{4}{|C|}} \qquad (\text{ i.e. } m^3 \stackrel{def.}{=} \tfrac{4}{C}), \\ n &\stackrel{def.}{=} \frac{B}{3C}, \qquad g_2 \stackrel{def.}{=} \frac{mB^2}{3C} = mnB, \\ g_3 &\stackrel{def.}{=} \frac{2B^3}{27C^2} - 2A, \qquad p \stackrel{def.}{=} -\frac{g_2}{4} = -\frac{mB^2}{12C}, \\ q &\stackrel{def.}{=} -\frac{g_3}{4} = -\frac{B^3}{54C^2} + \frac{A}{2}. \end{aligned} \tag{15}$$

The point of these definitions is that if $v(t)$ is the function given by $v(t) \stackrel{def.}{=} [y(t)-n]/m$, then equation (13) transforms to the differential equation

$$\frac{\left(mv + \frac{B}{3C}\right)^2 \dot{v}^2}{4(v^3 + pv + q)} = \frac{4}{m^2}, \tag{16}$$

as a direct computation reveals. Moreover, if $w(x)$ is an inverse function of $y(t)$ then $z(x) \stackrel{def.}{=} w(mx+n)$ is an inverse function of $v(t)$ so that from equation (16) the equation

$$z'(x)^2 = \frac{m^2}{16}\frac{\left(mx + \frac{B}{3C}\right)^2}{X(x)}, X(x) \stackrel{def.}{=} x^3 + px + q \tag{17}$$

is directly derived. This means that we can focus on solving

$$z'(x) = \pm\left[\frac{mB}{12C}\frac{1}{\sqrt{X(x)}} + \frac{m^2}{4}\frac{x}{\sqrt{X(x)}}\right], \tag{18}$$

which involves the elliptic integrals $\displaystyle\int \frac{dx}{\sqrt{X(x)}}, \int \frac{xdx}{\sqrt{X(x)}}$, that in turn involve a consideration of the roots of the cubic equation $X(x) = 0$ for their evaluation. The case of interest here is when $X(x) = 0$ has one real root r_1 and two complex roots r_2, r_3 - the other cases being simpler to deal with. Necessarily r_2 and r_3 are complex conjugates: $r_3 = \overline{r}_2$. The condition that $X(x) = 0$ indeed has a single real root r_1 and two complex conjugate roots r_2, r_3 is that its discriminant $\triangle = -4p^3 - 27q^2 \stackrel{def.}{=} \frac{1}{16}\left[g_2^3 - 27g_3^2\right] \stackrel{def.}{=} \frac{1}{16}\frac{4A}{C^2}\left[2B^3 - 27AC^2\right]$ (since $m^3 = \frac{4}{C}$) should be negative: $2B^3 - 27AC^2 < 0$ (given that $A > 0$), which we therefore assume.

Associated with the roots are useful parameters $\sigma, \rho, g, t_1, t_2$:

$$\sigma \overset{def.}{=} Imr_2, \quad \rho \overset{def.}{=} -r_1/2, \quad g \overset{def.}{=} \frac{1}{\sqrt[4]{9\rho^2+\sigma^2}},$$

$$t_1 \overset{def.}{=} r_1 + \sqrt{(\rho - r_1)^2 + \sigma^2} = r_1 + \sqrt{9\rho^2+\sigma^2}, \tag{19}$$

$$t_2 \overset{def.}{=} r_1 - \sqrt{(\rho - r_1)^2 + \sigma^2} = r_1 - \sqrt{9\rho^2+\sigma^2}.$$

Here $\sigma \neq 0$, since otherwise r_2 would be a second real root. The cubic $X(x)$ admits the factorization $X(x) = (x - r_1)[(x-\rho)^2 + \sigma^2]$, which shows that (since $\sigma \neq 0$) $X(x) > 0$ for $x > r_1$. Thus $x > r_1$ will be a convenient assumption in our discussion of functions like $X(x)$ and $z'(x)$ in (18), for example, where $\frac{1}{\sqrt{X(x)}}$ appears. The elliptic modulus k that will be employed in (14) is given by

$$k \overset{def.}{=} +\sqrt{\frac{\sqrt{\frac{9r_1^2}{4}+\sigma^2} - \frac{3r_1}{2}}{2\sqrt{\frac{9r_1^2}{4}+\sigma^2}}} = +\sqrt{\frac{\sqrt{9\rho^2+\sigma^2}+3\rho}{2\sqrt{9\rho^2+\sigma^2}}}. \tag{20}$$

We note that $0 < k < 1$. For $\sigma \neq 0$, $\sqrt{9\rho^2+\sigma^2} > \sqrt{9\rho^2} = 3|\rho| \geq \pm 3\rho \Rightarrow 2\sqrt{9\rho^2+\sigma^2} = \sqrt{9\rho^2+\sigma^2} + \sqrt{9\rho^2+\sigma^2} > \sqrt{9\rho^2+\sigma^2} + 3\rho > 0$ so that $1 > \frac{\sqrt{9\rho^2+\sigma^2}+3\rho}{2\sqrt{9\rho^2+\sigma^2}} > 0$; i.e. $1 > k^2 > 0 \Rightarrow 1 > k > 0$, as desired.

With the preceding notation and definitions in place, we can now construct a crucial function $u(x) = u(x,k)$ that facilitates the expression of solutions of equation (18). For

$$\begin{aligned} \theta(x) &\overset{def.}{=} 1 - \frac{(x-t_1)^2}{(x-t_2)^2} \\ &= \frac{[2x-(t_1+t_2)](t_1-t_2)}{(x-t_2)^2}, \quad r_1 < x, \end{aligned} \tag{21}$$

and for $EF(x,k)$ in (14) and k in (20), we set

$$u(x) = u(x,k) \overset{def.}{=} EF(\sqrt{\theta(x)}, k). \tag{22}$$

Here $t_2 < r_1 < x \Rightarrow x - t_2 \neq 0$ in (21). Also $\theta(t_1) = 1$, but for $r_1 < x \neq t_1$ (i.e. $(x-t_1)^2 > 0$) $\theta(x) < 1$. Moreover, $2x > 2r_1 = t_1 + t_2$ (by (19)) $\Rightarrow \theta(x) > 0$ (since $t_1 > t_2$). That is, $r_1 < x \neq t_1 \Rightarrow 0 < \theta(x) < 1 \Rightarrow 0 < \sqrt{\theta(x)} < 1$ in (22). Since $sn(y)$ is the inverse function of $EF(x)$, we can provide a second description of $u(x)$. Namely, $\sqrt{\theta(x)} = sn(EF(\sqrt{\theta(x)})) \doteq sn(u(x))$, and therefore $cn(u(x)) = \sqrt{1 - sn^2(u(x))} = \sqrt{1-\theta(x)} \overset{def.}{=} \frac{|x-t_1|}{(x-t_2)}$ (since (again) $x > t_2$ for $x > r_1$) $\Rightarrow$

$$u(x,k) = cn^{-1}\left(\frac{|x-t_1|}{x-t_2}, k\right), \quad r_1 < x. \tag{23}$$

In terms of the elliptic functions $EF(x,k), u(x,k)$ defined in (14) and (22) (or (23)), again with the elliptic modulus k specified in definition (20), equation (18) is solved as follows, for an integration constant z_0:

$$z(x) = \pm\left[\left(\frac{m^2}{2g} + \frac{m^2 g t_2}{4} + \frac{mgB}{12C}\right)u(x) - \frac{m^2}{2g}EF(u(x)) + \frac{m^2\sqrt{X(x)}}{2(x-t_2)}\right] + z_0 \tag{24}$$

for $r_1 < x < t_1$, and

$$z(x) = \pm\left[\left(-\frac{m^2}{2g} - \frac{m^2 g t_2}{4} - \frac{mgB}{12C}\right)u(x) + \frac{m^2}{2g}EF(u(x)) + \frac{m^2\sqrt{X(x)}}{2(x-t_2)}\right] + z_0 \tag{25}$$

for $t_1 < x$. m, g, t_1, t_2 are defined in (15), (19), and we assume that $A > 0$ in (13), and $27AC^2 > 2B^3$ so that r_1 is the unique real root of the cubic equation $X(x) = 0$, for $X(x)$ in (17) with p, q there also defined in (15).

Going back to the definition $z(x) \overset{def.}{=} w(mx+n)$ (for $n \overset{def.}{=} B/3C$ in (15)), where $w(x)$ is an inverse function of $y(t)$, we see that the initial differential equation (13) is solved implicitly by way of $w(x)$ given by

$$w(x) = z\left(\frac{x-n}{m}\right), \tag{26}$$

for $z(x)$ given by (24), or by (25).

IV. THE CORRESPONDENCES

The Lidsey correspondence between BEC's and cosmology originates by way of a comparison of equation (2) with equation (7), resulting in the following table.

TABLE I: BEC $\leftrightarrow$ FLRW correspondence

I_2	$\leftrightarrow$	a^2
I_3	$\leftrightarrow$	$2(aH)$
$I_3^2/4I_2$	$\leftrightarrow$	H^2
$\left[(d-1)(d-2)I_4 - \Lambda_d\right]/K_d$	$\leftrightarrow$	ρ_ϕ
$\left[(d-2)\omega^2 I_2 - (d-1)(d-2)I_4 + \Lambda_d\right]/K_d$	$\leftrightarrow$	p_ϕ

Similarly, the correspondence in the case of the Bianchi I cosmology originates by comparing equations (2) and (12) for which one obtains the following table.

TABLE II: BEC $\leftrightarrow$ Bianchi I correspondence

I_2	$\leftrightarrow$	$R^{2(d-1)}$
I_3	$\leftrightarrow$	$2(d-1)(R^{(d-1)}H_R)$
$I_3^2/4I_2$	$\leftrightarrow$	$(d-1)^2 H_R^2$
$\left[\frac{(d-2)}{(d-1)}I_4 - \Lambda_d\right]/K_d$	$\leftrightarrow$	ρ_ϕ
$\left[\frac{(d-2)}{(d-1)}\omega^2 I_2 - \frac{(d-2)}{(d-1)}I_4 + \Lambda_d\right]/K_d$	$\leftrightarrow$	p_ϕ
λ	$\leftrightarrow$	$-2(d-1)K_d D/(d-2)$

V. SOME EXAMPLES

The moments are known to satisfy the equations [12, 13]

$$\begin{aligned} \dot{I}_1(t) &= 0, \quad \dot{I}_2(t) = I_3(t), \\ \dot{I}_3(t) &= -2\omega(t)^2 I_2(t) + 4I_4(t), \quad \dot{I}_4(t) = \frac{-\omega(t)^2}{2} I_3(t). \end{aligned} \tag{27}$$

At this point we assume an equation of state $p_\phi = (\gamma-1)\rho_\phi$, with $\gamma > 0$. We indicate how to extend the discussion in section IV of [12]. The initial step is to equate the fifth BEC entry in Table I with $(\gamma-1)$ times the forth entry there and solve for ω^2. The result is that

$$\omega^2 = \gamma(d-1)\frac{I_4}{I_2} - \frac{\gamma\Lambda_d}{(d-2)I_2}, \tag{28}$$

which by (27) gives (since $I_3 = \dot{I}_2$)

$$\dot{I}_4 = \left[-\frac{\gamma(d-1)}{2}\frac{I_4}{I_2} + \frac{\gamma\Lambda_d}{2(d-2)I_2}\right]\dot{I}_2. \tag{29}$$

(29) being a first order, linear differential equation consequently has the solution

$$I_4 = \frac{\alpha}{I_2^{\gamma(d-1)/2}} + \frac{\Lambda_d}{(d-1)(d-2)}, \tag{30}$$

for an integration constant α, which plugged into equation (28) yields the Λ_d-independent result

$$\omega^2 = \frac{\gamma(d-1)\alpha}{I_2^{[\gamma(d-1)+2]/2}}. \tag{31}$$

By equations (1) and (27)

$$\lambda = 2I_2 I_4 - I_3^2/4 = 2I_2 I_4 - \dot{I}_2^2/4, \tag{32}$$

which with the help of equation (30) can be re-written as

$$\frac{\dot{I}_2^2}{4} = \frac{2\alpha}{I_2^{[\gamma(d-1)-2]/2}} + \frac{2\Lambda_d I_2}{(d-1)(d-2)} - \lambda. \tag{33}$$

Note that the above assumption that $\gamma > 0$ (so in particular, $\gamma \neq 0$) rules out the un-wanted conclusion $\omega = 0$, by equation (31) (or (28)), and also the conclusion that $I_4(t)$ is a constant function, by equation (29). Equations (31) and (33) govern the time-dependent trapping frequency and thus the external potential $V(r,t) = \omega(t)^2 r^2/2$ also, although equation (33) is a bit complicated. If $\Lambda_d = 0$, for example, it has the implicit solution

$$\frac{I_2^{q/2+1}\, {}_2F_1\left(\frac{1}{2}+\frac{1}{q}, \frac{1}{2}; \frac{3}{2}+\frac{1}{q}; \frac{\lambda I_2^q}{2\alpha}\right)}{\sqrt{2\alpha}(q+2)} = \pm t + t_0 \tag{34}$$

for $q \overset{def.}{=} [\gamma(d-1)-2]/2$, and for an integration constant t_0.

Consider the choice $\gamma = 6/(d-1)$, for example, which corresponds to a stiff perfect fluid. Then equation (33) becomes

$$\frac{\dot{I}_2^2}{4} = \frac{2\alpha}{I_2^2} - \lambda + \frac{2\Lambda_d I_2}{(d-1)(d-2)}. \tag{35}$$

This is equation (13) for $A \overset{def.}{=} \alpha, B \overset{def.}{=} \lambda$, and $C \overset{def.}{=} 2\Lambda_d/(d-1)(d-2)$, with $\Lambda_d \neq 0$, where we note that $A = \alpha > 0$ by (30), since $\gamma > 0$. Equation (35) can therefore be solved (implicitly) in terms of the elliptic functions $EF(x,k), E(u,k)$ in definition (14). Namely, the inverse function of $I_2(t)$ is given by $w(x) = z\left(\frac{x-n}{m}\right)$ where $z(x)$ is given by equation (24) or (25), according to equation (26) and the notation of section III. The hypothesis there is that $27AC^2 > 2B^3$. This means that $54\alpha\Lambda_d^2/(d-1)^2(d-2)^2 > \lambda^3$, which as we have seen corresponds to the condition of a negative discriminant Δ. The cases $\Delta > 0$ and $\Delta = 0$ are treated in the Appendix. By equation (31)

$$\omega^2 = \frac{6\alpha}{I_2^4}. \tag{36}$$

In case $\Lambda_d = 0$, one can write equation (33) as $\frac{1}{2}I_2 dI_2/\sqrt{2\alpha - \lambda I_2^2} = \pm dt$, which integrated gives $-\frac{1}{2\lambda}\sqrt{2\alpha - \lambda I_2^2} = \pm t + t_0$, or

$$I_2^2(t) = \frac{2\alpha}{\lambda} - 4\lambda\left[\pm t + t_0\right]^2, \tag{37}$$

for an integration constant t_0. By equation (37), equation (36) is explicated.

Another choice of interest is $\gamma = \frac{4}{d-1}$, (for a universe dominated by matter when $d = 4$) in which case equation (33) reads $\dot{I}_2^2/4 = A/I_2 - B + CI_2$, or

$$\int \frac{\sqrt{I_2}}{\sqrt{CI_2^2 - BI_2 + A}} dI_2 = \pm 2t + t_0 \tag{38}$$

where $A = 2\alpha, B = \lambda, C = 2\Lambda_d/(d-1)(d-2)$, and $t_0 =$ an integration constant. For a general value of C the integral in (38) can be expressed explicitly in terms of the elliptic functions $EF(x,k), E(u,k)$ of the first and second kind in

definition (14) by use of Maple, for example. Thus again for $\Lambda_d \neq 0, I_2(t)$ can be determined implicitly, and moreover, by equation (31),

$$\omega^2 = \frac{4\alpha}{I_2^3}. \tag{39}$$

In the particular (easier) case when $\Lambda_d = 0$ (i.e. $C = 0$), for example, the integral in (38) is an elementary function. Namely, equation (38) reduces to the equation

$$-\frac{\sqrt{A - BI_2}\sqrt{I_2}}{B} + \frac{A}{B^{3/2}} arctan\left(\sqrt{\frac{BI_2}{A - BI_2}}\right) = \pm 2t + t_0, \tag{40}$$

again for $A \overset{def.}{=} 2\alpha > 0, B \overset{def.}{=} \lambda$, say for $0 < I_2(t) < \frac{A}{B} = \frac{2\alpha}{\lambda}$, where already $I_2(t) > 0$ and (as we have seen) $\alpha > 0$. Thus one needs that $\lambda > 0$; in [12] the choice $\lambda = 1$ is made.

As a final example, regarding table I, we take $\gamma = \frac{3}{d-1}$. Then equation (33) can be written as

$$\int \frac{I_2^{1/4} dI_2}{\sqrt{2\alpha + CI^{3/2} - \lambda I_2^{1/2}}} = \pm 2t + t_0, \tag{41}$$

again for $C \overset{def.}{=} 2\Lambda_d/(d-1)(d-2)$. However, the integral here is non-tractable unless $C = 0$ (i.e. $\Lambda_d = 0$), in which case its evaluation gives the implicit equation

$$\frac{6\alpha^2}{\lambda^{5/2}} arctan\left(\frac{\sqrt{\lambda} I_2^{1/4}}{\sqrt{2\alpha - \lambda I_2^{1/2}}}\right) - \sqrt{2\alpha - \lambda I_2^{1/2}}\left[\frac{3\alpha I_2^{1/4}}{\lambda^2} + \frac{I_2^{3/4}}{\lambda}\right] = \pm 2t + t_0 \tag{42}$$

for $I_2(t)$. Also for $\gamma = 3/(d-1), \omega^2 = 3\alpha/I_2^{5/2}$ by equation (31).

To close things out we present a few examples regarding Table II for a Bianchi I cosmology, where we maintain the equation of state $p_\phi = (\gamma - 1)\rho_\phi, \gamma > 0$. In place of equations (28) and (29), one quickly checks that the equations $\omega^2 = \gamma I_4/I_2 - \gamma(d-1)\Lambda_d/(d-2)I_2, \dot{I}_4 + (\gamma/2)\dot{I}_2 I_4/I_2 = (\gamma/2)(d-1)\Lambda_d \dot{I}_2/(d-2)I_2$ follow by Table II. The latter equation has solution

$$I_4 = \frac{\alpha}{I_2^{\gamma/2}} + \left(\frac{d-1}{d-2}\right)\Lambda_d \tag{43}$$

by which the former equation can be written as

$$\omega^2 = \frac{\gamma\alpha}{I_2^{\gamma/2+1}}, \tag{44}$$

which again is Λ_d-independent, and where (again) α is an integration constant. By equations (32) and (43) we deduce that

$$\frac{\dot{I}_2^2}{4} = \frac{2\alpha}{I_2^{\gamma/2-1}} + 2\left(\frac{d-1}{d-2}\right)\Lambda_d I_2 - \lambda \tag{45}$$

is the Bianchi I version of equation (33). The equation

$$\int \frac{dI_2}{\sqrt{2\alpha I_2^{1-\gamma/2} + bI_2 - \lambda}} = \pm 2t + t_0 \tag{46}$$

is a re-expression of equation (45) for $b \overset{def.}{=} 2(d-1)\Lambda_d/(d-2)$.

An obvious solution for I_2 is obtained by choosing $\gamma = 2$, for example, in which case equation (46) reads (for $\Lambda_d \neq 0$) $(2/b)\sqrt{bI_2(t) - \lambda + 2\alpha} = \pm 2t + t_0$, or

$$I_2(t) = \frac{\Lambda_d}{2}\left(\frac{d-1}{d-2}\right)(\pm 2t + t_0)^2 + \frac{(\lambda - 2\alpha)}{2\Lambda_d}\left(\frac{d-2}{d-1}\right). \tag{47}$$

Equation (44) then assumes the explicit form $\omega^2(t) = 2\alpha/I_2^2(t)$.

As a second example, choose $\gamma = 1$, $\Lambda_d = 0$ (i.e. $b = 0$). Then equations (44) and (45) read

$$\frac{2\sqrt{2\alpha I_2^{1/2}(t) - \lambda}\,\left(\alpha I_2^{1/2}(t) + \lambda\right)}{3\alpha^2} = \pm 2t + t_0$$

$$\omega^2(t) = \alpha/I_2^{3/2}(t), \tag{48}$$

since $\int \frac{dx}{\sqrt{a\sqrt{x} - c}} = 4\sqrt{a\sqrt{x} - c}\,(a\sqrt{x}+2c)/3a^2$. More generally, the integral $\int \frac{dx}{\sqrt{ax^{1/(n+1)} - c}}$ for $n = 0, 1, 2, 3\cdots$, which normally involves the hypergeometric function ${}_2F_1$, can be explicitly computed. Thus for $\Lambda_d = 0$, we obtain a family of examples by choosing $\gamma = \gamma_n \stackrel{def.}{=} 2\left(1 - \frac{1}{(n+1)}\right), n = 0, 1, 2, 3 \ldots$. For the record, we are able to derive the general formula for the corresponding integral in (46),

$$\int \frac{dx}{\sqrt{ax^{1/(n+1)} - c}} = \frac{2(n+1)}{a^{n+1}}\sqrt{ax^{1/(n+1)} - c}\sum_{j=0}^{n} \frac{n!(ax^{1/(n+1)} - c)^{n-j}c^j}{j!(n-j)!(2n-2j+1)}. \tag{49}$$

The case $n = 1$, was just treated. If $n = 2$, for example, $\gamma = \gamma_3 = 4/3$ and the corresponding integral in equation (46) is computed by the formula

$$\int \frac{dx}{\sqrt{ax^{1/3} - c}} = \frac{2\sqrt{ax^{1/3} - c}}{5a^3}\left[3a^2x^{2/3} + 4acx^{1/3} + 8c^2\right]. \tag{50}$$

If $n = 3$, then $\gamma = \gamma_3 = 3/2$ and the integral in equation (46) is computed by

$$\int \frac{dx}{\sqrt{ax^{1/4} - c}} = \frac{8}{35a^4}\sqrt{ax^{1/4} - c}\left[5a^3x^{3/4} + 6a^2x^{1/2}c + 8ax^{1/4}c^2 + 16c^3\right]. \tag{51}$$

In general for $\gamma = \gamma_n$, equation (44) shows that

$$\omega^2 = \frac{2n\alpha}{(n+1)I_2^{2-1/(n+1)}}. \tag{52}$$

APPENDIX: COMPUTATION OF SOME ELLIPTIC INTEGRALS

The problem of solving equation (13) has been reduced to that of solving equation (18), which in turn is a matter of computation of the elliptic integrals $I_j(x) \stackrel{def.}{=} \int \frac{x^j dx}{\sqrt{X(x)}}$. For the reader's convenience we provide the result, which can be deduced from formulas in [3] coupled with a few extra arguments. We use freely the notation of section III.

There are three cases: $\Delta < 0, \Delta > 0$, and $\Delta = 0$. First assume that $\Delta < 0$. That is, $27AC^2 > 2B^3$ so that $X(x) = 0$ has a single real root r_1. Then, omitting integration constants, we have

$$I_0(x) = \begin{bmatrix} gu(x) & \text{for } r_1 < x < t_1 \\ -gu(x) & \text{for } t_1 < x \end{bmatrix}, \tag{A1}$$

$$I_1(x) = \begin{bmatrix} \frac{2}{g}\left(u(x) - E(u(x)) + \frac{g\sqrt{X(x)}}{x - t_2}\right) + t_2 gu(x) & \text{for } r_1 < x < t_1 \\ -\frac{2}{g}\left(u(x) - E(u(x)) - \frac{g\sqrt{X(x)}}{x - t_2}\right) - t_2 gu(x) & \text{for } t_1 < x \end{bmatrix}$$

for $u(x) = u(x, k)$ in definition (22).

If $\Delta > 0$, i.e. $27AC^2 < 2B^2$, then $X(x) = 0$ has three distinct *real* roots a, b, c, say $a > b > c$. In this case we now define $k, u(x)$ by

$$k \overset{def.}{=} \sqrt{\frac{b-c}{a-c}},\ u(x) = u(x,k) \overset{def.}{=} EF\left(\sqrt{\frac{x-a}{x-b}}, k\right). \tag{A2}$$

Then $X(x) = (x-a)(x-b)(x-c)$ and for $x > a$

$$I_0(x) \ = \ \frac{2u(x)}{\sqrt{a-c}}, \tag{A3}$$

$$I_1(x) = 2\sqrt{a-c}\,[dn\ u(x)tn\ u(x) - E(u(x))] + \frac{2au(x)}{\sqrt{a-c}},$$

where $tn\ x \overset{def.}{=} snx/cnx$ as usual.

If $\Delta = 0$ (the final case), then $X(x) = 0$ has at least two real roots: $X(x) = (x-a)^2(x-c)$ for real numbers a, c. Thus the $I_j(x)$ are elementary functions computable by a calculus table of integrals, depending on whether $c = a$ or $c \neq a$.

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